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Some Sharp Inequalities for Algebraic Polynomials

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Let H_n be the set of all algebraic polynomials with real coefficients of degree at most $n (n+1 \in N)$. For $Q \in H_n$, $\alpha, \beta > -1$ we set

$$\|Q\|_{\alpha, \beta} := \left\{ \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} Q^2(x) \, dx \right\}^{1/2}.$$

Let $n+1 \in N$,

$$\varphi(n; \alpha, \beta; \gamma, \delta) := \sup \{ \|Q\|_{\alpha, \beta} : Q \in H_n, \|Q\|_{\gamma, \delta} = 1 \}.$$

In this paper we find explicit expressions in terms of n for $\varphi(n; \alpha, \beta; \alpha+1, \beta)$, $\varphi(n; \alpha+1, \beta; \alpha, \beta)$ in cases $|\alpha| = |\beta| = \frac{1}{2}$, and for $\varphi(n; \frac{3}{2}, \frac{3}{2}; \frac{1}{2}, \frac{1}{2})$, $\varphi(n; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2})$.

1. INTRODUCTION

For two sequences $\{\alpha_n\}_0^{\infty}$, $\{\beta_n\}_0^{\infty}$ of positive numbers, we shall write $\alpha_n \sim \beta_n$ if there exist constants $A_1, A_2 > 0$, independent of n, such that

$$A_1 \beta_n \leqslant \alpha_n \leqslant A_2 \beta_n \qquad (n+1 \in N).$$

The following relations were proved in [2]: if $\alpha, \beta, \gamma, \delta > -1$, then

$$\varphi(n; \alpha, \beta; \gamma, \delta) \sim \begin{cases} (n+1)^{\max(\gamma-\alpha, \delta-\beta)} & \text{if} \quad \gamma > \alpha \quad \text{or} \quad \delta > \beta, \\ 1 & \text{if} \quad \gamma \leqslant \alpha \quad \text{and} \quad \delta \leqslant \beta. \end{cases} \tag{1}$$

The following statement was proved in [6]: let $\varepsilon, \mu > 0$, $-1+\varepsilon < \alpha$, $\gamma < \mu$. There exist constants $C_1, C_2 > 0$, depending only on ε and μ, such that $\forall n \in N$ we have

$$C_1 \cdot n^{\gamma - (\alpha/2) + (1/2)} (n+1)^{-(\alpha/2) - (1/2)} \leqslant \varphi(n; \alpha, \alpha; \gamma, \gamma) \leqslant C_2 \cdot n^{\gamma - (\alpha/2) + (1/2)} (n+1)^{-(\alpha/2) - (1/2)}.$$

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We note that explicit expressions for $\varphi(n; \alpha, \beta; \gamma, \delta)$ have not been found before. The search for exact constants in different kind of inequalities is and, we hope, always will be of considerable interest for mathematicians. However, it is not the exact constants themselves that are very important but the methods used for their determination, which are quite often interesting and instructive.

2. SOME CONSEQUENCES OF THE GAUSS–JACOBI MECHANICAL QUADRATURE

Let $w(x)$ be a weight function defined on $[-1, 1]$, i.e., Lesbesgue-measurable, nonnegative, and such that $\int_{-1}^{1} w(x)\,dx > 0$. Let $\{p_n(x)\}_0^\infty$ be the system of algebraic polynomials orthonormal on $[-1, 1]$ with respect to $w(x)$. We denote by $x_n^{(k)}$ $(n \in N,\ k = 1, 2, ..., n)$ the zeroes of $p_n(x)$, arranged in decreasing order. For the Jacobi weight $w(x) = (1-x)^\alpha (1+x)^\beta$ $(\alpha, \beta > -1)$ we set $x_n^{(k)} = x_n^{(k); \alpha, \beta}$ $(n \in N,\ k = 1, 2, ..., n)$. In this section we will make use of the Gauss–Jacobi mechanical quadrature [8] (formula (3.4.1)): there exist positive numbers $\lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_n^{(n)}$ such that $\forall \varphi \in H_{2n-1}$ $(n \in N)$ we have

$$\int_{-1}^{1} \varphi(x)\, w(x)\, dx = \sum_{\nu=1}^{n} \lambda_\nu^{(n)} \varphi(x_n^{(\nu)}). \tag{2}$$

We will use formula (2) to obtain explicit expressions for $\varphi(n; \alpha, \beta; \alpha+1, \beta)$ and $\varphi(n; \alpha+1, \beta; \alpha, \beta)$ in the cases $|\alpha| = |\beta| = \frac{1}{2}$.

THEOREM 1. *Let $\alpha, \beta > -1$, $n+1 \in N$. Then*

$$\varphi(n; \alpha, \beta; \alpha+1, \beta) = (1 - x_{n+1}^{(1); \alpha, \beta})^{-1/2} \tag{3}$$

$$\varphi(n; \alpha+1, \beta; \alpha, \beta) = (1 - x_{n+1}^{(n+1); \alpha, \beta})^{1/2}. \tag{4}$$

Proof. Let $w(x) = (1-x)^\alpha (1+x)^\beta$, $Q \in H_n$. Applying (2) to $\varphi(x) = (1-x)\, Q^2(x)$, we obtain

$$\int_{-1}^{1} (1-x)^{\alpha+1} (1+x)^\beta Q^2(x)\, dx$$

$$= \sum_{\nu=1}^{n+1} \lambda_\nu^{(n+1)} (1 - x_{n+1}^{(\nu); \alpha, \beta})\, Q^2(x_{n+1}^{(\nu); \alpha, \beta}), \tag{5}$$

from where

$$\int_{-1}^{1} (1-x)^{\alpha+1} (1+x)^{\beta} Q^2(x)\, dx \leqslant (1-x_{n+1}^{(n+1); \alpha, \beta}) \sum_{\nu=1}^{n+1} \lambda_{\nu}^{(n+1)} Q^2(x_{n+1}^{(\nu); \alpha, \beta}). \tag{6}$$

Taking into account (2), we obtain

$$\sum_{\nu=1}^{n+1} \lambda_{\nu}^{(n+1)} Q^2(x_{n+1}^{(\nu); \alpha, \beta}) = \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} Q^2(x)\, dx. \tag{7}$$

It follows from (6) and (7) that

$$\int_{-1}^{1} (1-x)^{\alpha+1} (1+x)^{\beta} Q^2(x)\, dx \leqslant (1-x_{n+1}^{(n+1); \alpha, \beta}) \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} Q^2(x)\, dx. \tag{8}$$

We consider now the polynomial $Q_1 \in H_n$ such that $Q_1(x_{n+1}^{(\nu); \alpha, \beta}) = 0$ ($\nu = 1, 2, ..., n$), $Q_1(x_{n+1}^{(n+1); \alpha, \beta}) = 1$. If we apply (5) and (7) to Q_1, we obtain

$$\int_{-1}^{1} (1-x)^{\alpha+1} (1+x)^{\beta} Q_1^2(x)\, dx = \lambda_{n+1}^{(n+1)} (1-x_{n+1}^{(n+1); \alpha, \beta}),$$

$$\lambda_{n+1}^{(n+1)} = \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} Q_1^2(x)\, dx,$$

which imply

$$\int_{-1}^{1} (1-x)^{\alpha+1} (1+x)^{\beta} Q_1^2(x)\, dx = (1-x_{n+1}^{(n+1); \alpha, \beta}) \int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} Q_1^2(x)\, dx. \tag{9}$$

From (8) and (9) we obtain (4). Equality (3) can be proved in a similar manner. Thus, Theorem 1 is proved. ∎

In the case $|\alpha| = |\beta| = \frac{1}{2}$ there are well-known explicit expressions for $x_{n+1}^{(1); \alpha, \beta}$ and $x_{n+1}^{(n+1); \alpha, \beta}$. In the other cases one can use numerous estimates for $x_{n+1}^{(1); \alpha, \beta}$ and $x_{n+1}^{(n+1); \alpha, \beta}$ [3, 4, 7, 8, pp. 121, 124, 138], yielding estimates from above for $\varphi(n; \alpha, \beta; \alpha+1, \beta)$ and $\varphi(n; \alpha+1, \beta; \alpha, \beta)$, where instead of $(1-x_{n+1}^{(1); \alpha, \beta})^{-1/2}$ and $(1-x_{n+1}^{(n+1); \alpha, \beta})^{1/2}$ there will be explicit functions of n, α, β.

Now we intend to derive some new inequalities for algebraic polynomials by making use of formula (2). Let $R_l \in H_l$ ($l \in N$), $|R_l(x)| \leqslant 1$ on $[-1, 1]$. It follows from (2) that $\forall Q \in H_n$ we have

$$\frac{\int_{-1}^{1} R_l(x)\, w(x)\, Q^2(x)\, dx}{\int_{-1}^{1} w(x)\, Q^2(x)\, dx} = \frac{\sum_{\nu=1}^{n+1+\lfloor l/2 \rfloor} \lambda_\nu^{(n+1+\lfloor l/2 \rfloor)} Q^2(x_{n+1+\lfloor l/2 \rfloor}^{(\nu)})\, R_l(x_{n+1+\lfloor l/2 \rfloor}^{(\nu)})}{\sum_{\nu=1}^{n+1+\lfloor l/2 \rfloor} \lambda_\nu^{(n+1+\lfloor l/2 \rfloor)} Q^2(x_{n+1+\lfloor l/2 \rfloor}^{(\nu)})} \tag{10}$$

From (10) we derive that

$$\sup \left\{ \int_{-1}^{1} R_l(x)\, w(x)\, Q^2(x)\, dx \colon Q \in H_n, \int_{-1}^{1} w(x)\, Q^2(x)\, dx = 1 \right\} \leqslant \max \left\{ R_l(x_{n+1+\lfloor l/2 \rfloor}^{(\nu)}), \nu = 1, 2, ..., n+1+\left\lfloor \frac{l}{2} \right\rfloor \right\}. \tag{11}$$

It follows directly from (11) that $\forall Q \in H_n$ the following estimate holds:

$$\int_{-1}^{1} (1 + R_l(x))\, w(x)\, Q^2(x)\, dx \leqslant \left(1 + \max \left\{ R_l(x_{n+1+\lfloor l/2 \rfloor}^{(\nu)}) \colon \nu = 1, 2, ..., n+1+\left\lfloor \frac{l}{2} \right\rfloor \right\} \right) \times \int_{-1}^{1} w(x)\, Q^2(x)\, dx. \tag{12}$$

Similarly, $\forall Q \in H_n$ we have

$$\int_{-1}^{1} (1 + R_l(x))\, w(x)\, Q^2(x)\, dx \geqslant \left(1 + \min \left\{ R_l(x_{n+1+\lfloor l/2 \rfloor}^{(\nu)}) \colon \nu = 1, 2, ..., n+1+\left\lfloor \frac{l}{2} \right\rfloor \right\} \right) \times \int_{-1}^{1} w(x)\, Q^2(x)\, dx. \tag{13}$$

If $l \geqslant 2$, then, in general, in (12) and (13) we cannot replace the inequality signs $\leqslant$ and $\geqslant$ by the equality sign. In fact, we are unable to use the same line of reasoning we used before to prove (3) and (4): there is no $Q \in H_n$ such that $Q(x_{n+1+\lfloor l/2 \rfloor}^{(\nu)}) = 0$ for $\nu = 2, ..., n+1+\lfloor l/2 \rfloor$, $Q(x_{n+1+\lfloor l/2 \rfloor}^{(1)}) = 1$;

similarly, there is no $Q \in H_n$ such that $Q(x^{(\nu)}_{n+1+\lfloor l/2 \rfloor}) = 0$ for $\nu = 1, 2, ..., n + \lfloor l/2 \rfloor$, $Q(x^{n+1+\lfloor l/2 \rfloor}_{n+1+\lfloor l/2 \rfloor}) = 1$. We can now conclude that, in general, inequality (12) is not sharp, i.e. the constant $1 + \max\{ R_l(x^{(\nu)}_{n+1+\lfloor l/2 \rfloor})$: $\nu = 1, 2, ..., n + 1 + \lfloor l/2 \rfloor\}$ need not be the smallest on the whole class $Q \in H_n$. The same is true for the inequality (13).

Now we consider the particular case $w(x) = (1 - x^2)^{1/2}$; then $x^{(\nu)}_n = \cos(\nu\pi/n + 1)$ $(\nu = 1, 2, ..., n)$. We set $R_l(x) = -x^l$, $l = 2k$, $k \in N$. It is easy to derive from (13) that $\forall Q \in H_n$ we have

$$\int_{-1}^{1} (1 - x^{2k})(1 - x^2)^{1/2} Q^2(x)\, dx \geqslant \left(1 - \cos^{2k} \frac{\pi}{n + k + 2}\right) \int_{-1}^{1} (1 - x^2)^{1/2} Q^2(x)\, dx. \tag{14}$$

We will prove later in this paper that for $k = 1$ the inequality (14) is sharp.

By applying (12) to the same particular case we obtain that $\forall Q \in H_n$ we have

$$\int_{-1}^{1} (1 - x^{2k})(1 - x^2)^{1/2} Q^2(x)\, dx \leqslant \left(1 - \cos^{2k} \frac{(n + k + 1)\, \pi}{2(n + k + 2)}\right) \int_{-1}^{1} (1 - x^2)^{1/2} Q^2(x)\, dx \tag{15}$$

if $n + k + 1$ is even. We will prove below that for n even and $k = 1$ the inequality (15) is sharp.

We note that B. Bojanov [1] applied the Gauss–Jacobi mechanical quadrature to prove certain Duffin–Schaeffer type inequalities.

3. THE MAIN THEOREM

THEOREM 2. *If $n + 1 \in N$, then*

$$\varphi\left(n; \frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}\right) = \left(\sin \frac{\pi}{n + 3}\right)^{-1}, \tag{16}$$

$$\varphi\left(n; \frac{3}{2}, \frac{3}{2}; \frac{1}{2}, \frac{1}{2}\right) = \begin{cases} \cos \dfrac{\pi}{2(n + 3)} & \text{if } n \text{ is even,} \\[2ex] \cos \dfrac{\pi}{2(n + 2)} & \text{if } n \text{ is odd.} \end{cases} \tag{17}$$

4. SOME AUXILIARY STATEMENTS

We need two algebraic lemmas to carry out the proof of Theorem 2.

LEMMA 1. *The following equality holds*:

$$\begin{vmatrix} a_{11} & 0 & a_{12} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} & 0 & \cdots & 0 & 0 & 0 & 0 \\ a_{21} & 0 & a_{22} & 0 & a_{23} & \cdots & 0 & 0 & 0 & 0 \\ 0 & b_{21} & 0 & b_{22} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{32} & 0 & a_{33} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{n-1,n-1} & 0 & a_{n-1,n} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_{n-1,n-1} & 0 & b_{n-1,n} \\ 0 & 0 & 0 & 0 & 0 & \cdots & a_{n,n-1} & 0 & a_{n,n} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & b_{n,n-1} & 0 & b_{n,n} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & 0 \\ a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 \\ 0 & a_{32} & a_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & a_{n,n-1} & a_{n,n} \end{vmatrix} \cdot \begin{vmatrix} b_{11} & b_{12} & 0 & \cdots & 0 & 0 \\ b_{21} & b_{22} & b_{23} & \cdots & 0 & 0 \\ 0 & b_{32} & b_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & 0 & \cdots & b_{n,n-1} & b_{n,n} \end{vmatrix} \tag{18}$$

Proof. It is easy to prove (18) by expanding the determinant on the left-hand side of (18) over the rows 1, 3, ..., $2n-1$ with the aid of Laplace's theorem. ∎

LEMMA 2. *Let*

$$\Delta_n(\lambda) = \begin{vmatrix} 2\lambda-2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 2\lambda-2 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & 0 & 2\lambda-2 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2\lambda-2 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 2\lambda-2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2\lambda-2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 2\lambda-2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 2\lambda-2 \end{vmatrix}$$

be a determinant of order n, $n \in N$. *If* $0 < \lambda < 2$, $\theta = \arccos(\lambda - 1)$, *then we have*

$$\Delta_{2m}(\lambda) = \frac{\sin^2(m+1)\,\theta}{\sin^2\theta}, \qquad m \in N; \tag{19}$$

$$\Delta_{2m+1}(\lambda) = \frac{\sin(m+2)\,\theta \sin(m+1)\,\theta}{\sin^2\theta}, \qquad m+1 \in N. \tag{20}$$

Proof. First we prove (19). We denote

$$\Delta_{m,0}(\lambda) = \begin{vmatrix} 2\lambda-2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2\lambda-2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2\lambda-2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2\lambda-2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2\lambda-2 \end{vmatrix};$$

$\Delta_{m,0}(\lambda)$ is a determinant of order m. Making use of (18), we obtain

$$\Delta_{2m}(\lambda) = (\Delta_{m,0}(\lambda))^2. \tag{21}$$

We apply mathematical induction to prove that

$$\Delta_{m,0}(\lambda) = \frac{\sin(m+1)\,\theta}{\sin\theta}, \qquad m \in N. \tag{22}$$

For $m = 1$ and $m = 2$ relation (22) is obvious since

$$\Delta_{1,0}(\lambda) = 2\lambda - 2 = 2\cos\theta = \frac{\sin 2\theta}{\sin\theta},$$

$$\Delta_{2,0}(\lambda) = 4(\lambda-1)^2 - 1 = 4\cos^2\theta - 1 = \frac{\sin 3\theta}{\sin\theta}.$$

Assume that (22) holds for m, $m+1 \in N$. We have to show that under this assumption we have

$$\Delta_{m+2,0}(\lambda) = \frac{\sin(m+3)\,\theta}{\sin\theta}. \tag{23}$$

Expanding $\Delta_{m+2,0}(\lambda)$ with respect to the first column and making use of the induction hypothesis, we obtain

$$\begin{aligned}\Delta_{m+2,0}(\lambda) &= 2\cos\theta\cdot\Delta_{m+1,0}(\lambda)-\Delta_{m,0}(\lambda)\\ &= 2\cos\theta\cdot\frac{\sin(m+2)\,\theta}{\sin\theta}-\frac{\sin(m+1)\,\theta}{\sin\theta}\\ &= \frac{\sin(m+3)\,\theta+\sin(m+1)\,\theta-\sin(m+1)\,\theta}{\sin\theta}\\ &= \frac{\sin(m+3)\,\theta}{\sin\theta}.\end{aligned}$$

Therefore, since equality (23) is proved, so is (22). Equality (19) follows directly from (21) and (22).

The proof of (20) is more involved. First we prove by induction that for $m+1\in N$ we have

$$\Delta_{2m+1}(\lambda)=\frac{2\lambda-2}{\sin^2\theta}\sum_{k=0}^{m}(-1)^k\sin^2(m+1-k)\,\theta. \tag{24}$$

First we verify (24) for $m=0$ and $m=1$. For $m=0$ we have

$$\Delta_{2m+1}(\lambda)=\Delta_1(\lambda)=2\lambda-2=2\cos\theta,$$

$$\begin{aligned}\frac{2\lambda-2}{\sin^2\theta}\cdot\sum_{k=0}^{m}(-1)^k\sin^2(m+1-k)\,\theta &= \frac{2\lambda-2}{\sin^2\theta}\cdot\sin^2\theta\\ &= 2\lambda-2=2\cos\theta,\end{aligned}$$

while for $m=1$ we have

$$\Delta_{2m+1}(\lambda)=\Delta_3(\lambda)=\begin{vmatrix}2\cos\theta & 0 & 1\\ 0 & 2\cos\theta & 0\\ 1 & 0 & 2\cos\theta\end{vmatrix}=8\cos^3\theta-2\cos\theta,$$

$$\begin{aligned}\frac{2\lambda-2}{\sin^2\theta}\cdot\sum_{k=0}^{1}(-1)^k\sin^2(2-k)\,\theta &= \frac{2\cos\theta}{\sin^2\theta}(\sin^2 2\theta-\sin^2\theta)\\ &= 8\cos^3\theta-2\cos\theta.\end{aligned}$$

Thus, (24) is verified for $m=0$ and for $m=1$. To complete the proof of (24) it is sufficient to prove that if (24) holds for $m-1$ $(m\in N)$, then it holds also for $m+1$. Expanding $\Delta_{2m+3}(\lambda)$ with respect to the first column, we obtain

$$\Delta_{2m+3}(\lambda)=(2\lambda-2)\,\Delta_{2m+2}(\lambda)$$

$$+\begin{vmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 2\lambda-2 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0\\ 1 & 0 & 2\lambda-2 & 0 & 1 & \cdots & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 2\lambda-2 & 0 & \cdots & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 2\lambda-2 & \cdots & 0 & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & 0 & \cdots & 2\lambda-2 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 2\lambda-2 & 0 & 1\\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 2\lambda-2 & 0\\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 2\lambda-2 \end{vmatrix}.$$

Expanding the last determinant of order $2m+2$ with respect to the first column, making use of (19) and of the induction hypothesis, we obtain

$$\begin{aligned}\Delta_{2m+3}(\lambda)&=(2\lambda-2)\,\Delta_{2m+2}(\lambda)-(2\lambda-2)\,\Delta_{2m}(\lambda)+\Delta_{2m-1}(\lambda)\\ &=(2\lambda-2)\cdot\frac{\sin^2(m+2)\,\theta}{\sin^2\theta}-(2\lambda-2)\cdot\frac{\sin^2(m+1)\,\theta}{\sin^2\theta}\\ &\quad+\frac{2\lambda-2}{\sin^2\theta}\sum_{k=0}^{m-1}(-1)^k\sin^2(m-k)\,\theta\\ &=\frac{2\lambda-2}{\sin^2\theta}\sum_{k=0}^{m+1}(-1)^k\sin^2(m+2-k)\,\theta.\end{aligned}$$

Thus, equality (24) is proved.

It remains to prove that for any m $(m+1\in N)$ we have

$$\frac{2\lambda-2}{\sin^2\theta}\sum_{k=0}^{m}(-1)^k\sin^2(m+1-k)\,\theta=\frac{\sin(m+2)\,\theta\cdot\sin(m+1)\,\theta}{\sin^2\theta} \tag{25}$$

We consider two cases: (1) m is even; (2) m is odd.

(1) m is even. By applying the formula $\sin^2 \alpha = (1 - \cos 2\alpha)/2$ to each term under the summation sign on the right-hand side of (24), we obtain

$$
\begin{aligned}
\Delta_{2m+1}(\lambda) &= \frac{2\lambda - 2}{\sin^2 \theta} \cdot \sum_{k=0}^{m} (-1)^k \frac{1 - \cos(2m + 2 - 2k)\,\theta}{2} \\
&= \frac{\lambda - 1}{\sin^2 \theta} \left\{ \sum_{k=0}^{m} (-1)^k + \sum_{k=0}^{m} (-1)^{k+1} \cos(2m + 2 - 2k)\,\theta \right\} \\
&= \frac{\lambda - 1}{\sin^2 \theta} \sum_{k=0}^{m+1} (-1)^k \cos 2k\theta \\
&= \frac{\lambda - 1}{\sin^2 \theta} \cdot \Re \sum_{k=0}^{m+1} (-1)^k e^{2ik\theta} \\
&= \frac{\lambda - 1}{\sin^2 \theta} \Re \frac{1 - e^{2i(m+2)\,\theta}}{1 + e^{2i\theta}} \\
&= \frac{\lambda - 1}{\sin^2 \theta} \Re \frac{1 - \cos 2(m+2)\,\theta - i \sin 2(m+2)\,\theta}{1 + \cos 2\theta + i \sin 2\theta} \\
&= \frac{\lambda - 1}{\sin^2 \theta} \cdot \Re \frac{\sin(m+2)\,\theta[\sin(m+2)\,\theta - i \cos(m+2)\,\theta]}{\cos \theta(\cos \theta + i \sin \theta)} \\
&= \frac{\lambda - 1}{\sin^2 \theta} \cdot \Re \frac{-i \sin(m+2)\,\theta[\cos(m+2)\,\theta + i \sin(m+2)\,\theta]}{\cos \theta(\cos \theta + i \sin \theta)} \\
&= \frac{\lambda - 1}{\sin^2 \theta} \cdot \Re \frac{-i \sin(m+2)\,\theta[\cos(m+1)\,\theta + i \sin(m+1)\,\theta]}{\cos \theta} \\
&= \frac{\sin(m+2)\,\theta \cdot \sin(m+1)\,\theta}{\sin^2 \theta}.
\end{aligned}
$$

(2) m is odd. Making use of the formula $\sin^2 \alpha = (1 - \cos 2\alpha)/2$, we obtain

$$
\begin{aligned}
\Delta_{2m+1}(\lambda) &= \frac{\lambda - 1}{\sin^2 \theta} \sum_{k=0}^{m} (-1)^{k+1} \cos(2m + 2 - 2k)\,\theta \\
&= \frac{\lambda - 1}{\sin^2 \theta} \Re \sum_{k=1}^{m+1} (-1)^{k+1} e^{2ik\theta}
\end{aligned}
$$

$$= \frac{\lambda - 1}{\sin^2 \theta} \Re \frac{e^{2i\theta}(1 - e^{2i(m+1)\theta})}{1 + e^{2i\theta}}$$

$$= \frac{\lambda - 1}{\sin^2 \theta} \Re \frac{e^{2i\theta} \sin(m+1)\theta \cdot (\sin(m+1)\theta - i\cos(m+1)\theta)}{\cos\theta \cdot e^{i\theta}}$$

$$= \frac{\lambda - 1}{\sin^2 \theta} \Re \frac{(-i)\sin(m+1)\theta \cdot e^{i\theta} \cdot e^{i(m+1)\theta}}{\cos\theta}$$

$$= \frac{\lambda - 1}{\sin^2 \theta} \Re \frac{(-i)\sin(m+1)\theta \cdot e^{i(m+2)\theta}}{\cos\theta}$$

$$= \frac{\sin(m+1)\theta \cdot \sin(m+2)\theta}{\sin^2\theta}.$$

Lemma 2 is completely proved. ∎

5. COMPLETION OF THE PROOF OF THEOREM 2

We denote by $\{J_k^{\alpha, \beta}\}_0^\infty$ the system of Jacobi polynomials, orthonormal on $[-1, 1]$ with the weight function $(1-x)^\alpha (1+x)^\beta$, $\alpha, \beta > -1$. Let $Q \in H_n$. We represent Q as

$$Q = \sum_{k=0}^{n} c_k J_k^{(1/2, 1/2)}, \tag{26}$$

which implies

$$\int_{-1}^{1} (1-x^2)^{1/2} Q^2(x)\, dx = \sum_{k=0}^{n} c_k^2. \tag{27}$$

Making use of the identities

$$(1-x^2)\, J_n^{(1/2, 1/2)}(x) = \frac{1}{2} J_n^{(-1/2, -1/2)}(x) - \frac{1}{2} J_{n+2}^{(-1/2, -1/2)}(x), \qquad n \in N,$$

$$(1-x^2)\, J_0^{(1/2, 1/2)}(x) = \frac{1}{\sqrt{2}} J_0^{(-1/2, -1/2)}(x) - \frac{1}{2} J_2^{(-1/2, -1/2)}(x),$$

and taking into account (26), after simple transformations we obtain

$$
\begin{aligned}
(1-x^2)\,Q(x) &= \sum_{k=0}^{n} c_k(1-x^2)\,J_k^{(1/2,\,1/2)}(x) \\
&= c_0\left(\frac{1}{\sqrt{2}}\,J_0^{(-1/2,\,-1/2)}(x)-\frac{1}{2}\,J_2^{(-1/2,\,-1/2)}(x)\right) \\
&\quad+\sum_{k=1}^{n} c_k\left[\frac{1}{2}\,J_k^{(-1/2,\,-1/2)}(x)-\frac{1}{2}\,J_{k+2}^{(-1/2,\,-1/2)}(x)\right] \\
&= c_0\,\frac{1}{\sqrt{2}}\,J_0^{(-1/2,\,-1/2)}(x)-\frac{c_0}{2}\,J_2^{(-1/2,\,-1/2)}(x) \\
&\quad+\frac{1}{2}\sum_{k=1}^{n} c_k J_k^{(-1/2,\,-1/2)}(x)-\frac{1}{2}\sum_{k=1}^{n} c_k J_{k+2}^{(-1/2,\,-1/2)}(x) \\
&= \frac{c_0}{\sqrt{2}}\,J_0^{(-1/2,\,-1/2)}(x)-\frac{c_0}{2}\,J_2^{(-1/2,\,-1/2)}(x) \\
&\quad+\frac{1}{2}\sum_{k=1}^{n} c_k J_k^{(-1/2,\,-1/2)}(x)-\frac{1}{2}\sum_{i=3}^{n+2} c_{i-2} J_i^{(-1/2,\,-1/2)}(x) \\
&= \frac{c_0}{\sqrt{2}}\,J_0^{(-1/2,\,-1/2)}(x)-\frac{c_0}{2}\,J_2^{(-1/2,\,-1/2)}(x) \\
&\quad+\frac{1}{2}\sum_{k=3}^{n} (c_k-c_{k-2})\,J_k^{(-1/2,\,-1/2)}(x) \\
&\quad+\frac{1}{2}\,c_1 J_1^{(-1/2,\,-1/2)}(x)+\frac{1}{2}\,c_2 J_2^{(-1/2,\,-1/2)}(x) \\
&\quad-\frac{1}{2}\,c_{n-1} J_{n+1}^{(-1/2,\,-1/2)}(x)-\frac{1}{2}\,c_n J_{n+2}^{(-1/2,\,-1/2)}(x) \\
&= \frac{c_0}{\sqrt{2}}\,J_0^{(-1/2,\,-1/2)}(x)+\frac{1}{2}\,c_1 J_1^{(-1/2,\,-1/2)}(x) \\
&\quad+\frac{1}{2}\sum_{k=2}^{n} (c_k-c_{k-2})\,J_k^{(-1/2,\,-1/2)}(x) \\
&\quad-\frac{1}{2}\,c_{n-1} J_{n+1}^{(-1/2,\,-1/2)}(x)-\frac{1}{2}\,c_n J_{n+2}^{(-1/2,\,-1/2)}(x). \qquad (28)
\end{aligned}
$$

From (28), Parseval's equality yields

$$\int_{-1}^{1} ((1-x^2)\, Q(x))^2 \frac{dx}{\sqrt{1-x^2}}$$

$$= \int_{-1}^{1} (1-x^2)^{3/2}\, Q^2(x)\, dx$$

$$= \frac{c_0^2}{2} + \frac{c_1^2}{4} + \frac{1}{4} \sum_{k=2}^{n} (c_k - c_{k-2})^2 + \frac{c_{n-1}^2}{4} + \frac{c_n^2}{4}$$

$$= \frac{c_0^2}{2} + \frac{c_1^2}{4} + \frac{1}{4}\left(\sum_{k=2}^{n} c_k^2 + \sum_{k=2}^{n} c_{k-2}^2 - 2 \sum_{k=2}^{n} c_k c_{k-2} \right) + \frac{c_{n-1}^2}{4} + \frac{c_n^2}{4}$$

$$= \frac{c_0^2}{2} + \frac{1}{4}\left(\sum_{k=1}^{n} c_k^2 + \sum_{k=0}^{n} c_k^2 \right) - \frac{1}{2} \sum_{k=0}^{n-2} c_k c_{k+2}$$

$$= \frac{3}{4} c_0^2 + \frac{1}{2} \sum_{k=1}^{n} c_k^2 - \frac{1}{2} \sum_{k=0}^{n-2} c_k c_{k+2}. \tag{29}$$

It follows from (29) and (27) that in order to conclude the proof of Theorem 2 it is sufficient to find the maximum and the minimum values of the quadratic form

$$f(c_0, c_1, ..., c_n) = \tfrac{3}{2} c_0^2 + \sum_{k=1}^{n} c_k^2 - \sum_{k=0}^{n-2} c_k c_{k+2}$$

over the unit sphere $\sum_{k=0}^{n} c_k^2 = 1$. It is well known that these are equal to the greatest and smallest roots of the equation

$$\bar{\Delta}_{n+1}(\lambda) = \begin{vmatrix} \frac{3}{2} - \lambda & 0 & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & -\frac{1}{2} & \cdots & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1-\lambda & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-\lambda & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2} & 0 & 1-\lambda \end{vmatrix} = 0, \tag{30}$$

respectively. The determinant in (30) has order $n+1$. It is well known that all the roots of the equation (30) are real. We prove that all roots λ of the equation (30) satisfy $0 < \lambda < 2$. First of all, it is obvious that $\forall Q \in H_n$, $Q \neq 0$ we have

$$f(c_0, c_1, ..., c_n) = 2 \int_{-1}^{1} (1-x^2)^{3/2} Q^2(x)\, dx$$

$$< 2 \int_{-1}^{1} (1-x^2)^{1/2} Q^2(x)\, dx = 2 \sum_{k=0}^{n} c_k^2,$$

which implies $\lambda < 2$. On the other hand, it follows from (1) (second line) that $\min\{\|Q\|_{3/2,\,3/2}\colon Q \in H_n,\ \|Q\|_{1/2,\,1/2} = 1\} \approx 1$, which implies $\lambda > 0$. Thus, when considering the equation (30), we can make use of the equalities (19) and (20).

We write the first and the third elements of the first column of $\bar{\Delta}_{n+1}(\lambda)$ as $\frac{1}{2} + (1-\lambda)$ and $0 + (-\frac{1}{2})$, respectively. Then we represent $\bar{\Delta}_{n+1}(\lambda)$ as

$$\bar{\Delta}_{n+1}(\lambda) = \tilde{\Delta}_{n+1}(\lambda) + \tfrac{1}{2}\tilde{\Delta}_n(\lambda)$$

where the determinant

$$\bar{\Delta}_k(\lambda) = \begin{vmatrix} 1-\lambda & 0 & -\frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1-\lambda & 0 & -\frac{1}{2} & \cdots & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 1-\lambda & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 1-\lambda & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1-\lambda & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 0 & \cdots & -\frac{1}{2} & 0 & 1-\lambda \end{vmatrix}$$

is of order k. It is easy to see that

$$\tilde{\Delta}_k(\lambda) = \frac{1}{(-2)^k} \Delta_k(\lambda), \qquad k \in N,$$

so that

$$\bar{\Delta}_{n+1}(\lambda) = \frac{1}{(-2)^{n+1}} [-\Delta_n(\lambda) + \Delta_{n+1}(\lambda)]. \tag{31}$$

We consider two cases: (1) $n = 2m (m+1 \in N)$; (2) $n = 2m+1 (m+1 \in N)$.

(1) Let $n = 2m (m+1 \in N)$. Taking into account (31), (19), and (20), we obtain

$$\bar{\Delta}_{n+1}(\lambda) = -\frac{1}{2^{n+1}}\left[-\Delta_{2m}(\lambda) + \Delta_{2m+1}(\lambda)\right]$$

$$= -\frac{1}{2^{2m+1}}\left[-\frac{\sin^2(m+1)\,\theta}{\sin^2\theta} + \frac{\sin(m+2)\,\theta\cdot\sin(m+1)\,\theta}{\sin^2\theta}\right]$$

$$= \frac{\sin(m+1)\,\theta}{2^{2m+1}\sin^2\theta}\left[\sin(m+1)\,\theta - \sin(m+2)\,\theta\right]$$

$$= -\frac{\sin\dfrac{\theta}{2}\cdot\sin(m+1)\,\theta\cdot\cos\dfrac{(2m+3)\,\theta}{2}}{2^{2m}\sin^2\theta}.$$

(2) Let $n = 2m+1 (m+1 \in N)$. In a similar way we obtain

$$\bar{\Delta}_{n+1}(\lambda) = \frac{\sin\dfrac{\theta}{2}\cdot\sin(m+2)\,\theta\cdot\cos\dfrac{(2m+3)\,\theta}{2}}{2^{2m+1}\sin^2\theta}.$$

Now we find the greatest and the smallest zeroes of $\bar{\Delta}_{n+1}(\lambda)$, separately for the case $n = 2m$ $(m+1 \in N)$ and $n = 2m+1$ $(m+1 \in N)$.

In the case $n = 2m$ $(m+1 \in N)$, the function $\bar{\Delta}_{n+1}(\lambda)$ is a polynomial of degree $2m+1$ in λ. Its zeroes are $1+\cos(k\pi/m+1)$ $(k = 1, 2, ..., m)$ and $1+\cos(\pi(2l+1)/2m+3)$ $(l = 0, 1, ..., m)$. The smallest zero of $\bar{\Delta}_{n+1}(\lambda)$ is $1+\cos(\pi(2m+1)/2m+3) = 1+\cos(\pi(n+1)/n+3) = 2\sin^2(\pi/n+3)$ and, therefore

$$\min\left\{f(c_0, c_1, ..., c_n)\colon \sum_{k=0}^{n} c_k^2 = 1\right\} = 2\sin^2\frac{\pi}{n+3},$$

implying (16). The largest zero of $\bar{\Delta}_{n+1}(\lambda)$ is $1+\cos(\pi/2m+3) = 2\cos^2(\pi/2(n+3))$ and, therefore

$$\max\left\{f(c_0, c_1, ..., c_n)\colon \sum_{k=0}^{n} c_k^2 = 1\right\} = 2\cos^2\frac{\pi}{2(n+3)},$$

implying (17) (case n even).

In the case $n = 2m+1$ $(m+1 \in N)$, the function $\bar{\Delta}_{n+1}(\lambda)$ is a polynomial of degree $2m+2$ in λ. Its zeroes are $1+\cos(k\pi/m+2)$ $(k = 1, 2, ..., m+1)$ and $1+\cos(\pi(2l+1)/2m+3)$ $(l = 0, 1, ..., m)$. The smallest zero of $\bar{\Delta}_{n+1}(\lambda)$ is $1+\cos(\pi(m+1)/m+2) = 1-\cos(\pi/m+2) = 2\sin^2(\pi/2m+4) = 2\sin^2(\pi/n+3)$, which implies (16). The largest zero of $\bar{\Delta}_{n+1}(\lambda)$ is $1+\cos(\pi/2m+3) = 2\cos^2(\pi/2(n+2))$ and, consequently, relation (17) (case n odd) holds. This concludes the proof of Theorem 2.

6. REMARKS

1. The idea of reducing certain extremal problems for polynomials to the problem of finding the maximum and minimum values of a quadratic form in n variables over the unit sphere has been previously used by P. Turán [9] to prove some sharp Markov inequalities in the L_2-metric with Hermite and Laguerre weight functions.

2. We give now two examples of how to derive estimates of the constants in some Bernstein type inequalities by combining the statements (3), (4), (17)–(19) and some results obtained by Daugavet and Rafalson [2] and by Guessab and Milovanovic [5].

(a) The following sharp inequality was proved in [2]. If $Q \in H_n$, $m+1 \in N$, $\mu > -\frac{1}{2}$, then

$$\|Q^{(m)}\|_{2\mu+m,\,2\mu+m} \leqslant \sqrt{\frac{n!\,\Gamma(n+4\mu+m+1)}{(n-m)!\,\Gamma(n+4\mu+1)}}\,\|Q\|_{2\mu,\,2\mu}. \tag{32}$$

For $\mu = 1/4$ we obtain

$$\|Q^{(m)}\|_{m+1/2,\,m+1/2} \leqslant \sqrt{\frac{\Gamma(n+m+2)}{(n-m)!\,(n+1)}}\,\|Q\|_{1/2,\,1/2}.$$

Making use of (16) we obtain

$$\|Q^{(m)}\|_{m+1/2,\,m+1/2} \leqslant \sqrt{\frac{\Gamma(n+m+2)}{(n-m)!\,(n+1)}} \left(\sin\frac{\pi}{n+3}\right)^{-1} \|Q\|_{3/2,\,3/2}; \tag{33}$$

in particular, for $m=1$ we have

$$\|Q'\|_{3/2,\,3/2} \leqslant \sqrt{n(n+2)} \left(\sin\frac{\pi}{n+3}\right)^{-1} \|Q\|_{3/2,\,3/2}. \tag{34}$$

(b) The following sharp inequality was proved in [5]. If $Q \in H_n$, $m+1 \in N$, $\alpha, \beta > -1$, then

$$\|Q^{(m)}\|_{m+\alpha,\,m+\beta} \leqslant \sqrt{\frac{n!\,\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\,\Gamma(n+\alpha+\beta+1)}}\,\|Q\|_{\alpha,\,\beta}. \tag{35}$$

If in (35) we set $\alpha = \beta = -\frac{1}{2}$, then we obtain

$$\|Q^{(m)}\|_{m-1/2,\,m-1/2} \leqslant \sqrt{\frac{n\Gamma(n+m)}{(n-m)!}}\,\|Q\|_{-1/2,\,-1/2}. \tag{36}$$

Taking into account the well known expressions for the zeroes of the polynomials $J_n^{(-1/2,\,-1/2)}$ $(n \in N)$, we derive from (3) that

$$\varphi\left(n; -\frac{1}{2}, -\frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\right) = \frac{1}{\sqrt{2}\sin\dfrac{\pi}{4(n+1)}}. \tag{37}$$

From (36) and (37) it follows that

$$\|Q^{(m)}\|_{m-1/2,\, m-1/2} \leqslant \sqrt{\frac{n\Gamma(n+m)}{(n-m)!}} \cdot \frac{1}{\sqrt{2}\sin\dfrac{\pi}{4(n+1)}} \|Q\|_{1/2,\, -1/2}; \quad (38)$$

in particular, for $m=1$ we have

$$\|Q'\|_{1/2,\, 1/2} \leqslant \frac{n}{\sqrt{2}\sin\dfrac{\pi}{4(n+1)}} \|Q\|_{1/2,\, -1/2}. \quad (39)$$

The reader will have no trouble in deriving estimates of the constants in other Bernstein type inequalities by using (32), (35), (3), (4), (16)–(17).

We note that, although most likely the constants in (33), (34), (38), (39) are not exact, these inequalities cannot be improved in terms of the order of growth with respect to n. This assertion has been proved in [2].

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REFERENCES

1. B. Bojanov, An inequality of Duffin and Schaeffer type, *East J. Approx.* **1**(1) (1995), 37–46.
2. I. K. Daugavet and S. Z. Rafalson, Some inequalities of Markov–Nikolskii type for algebraic polynomials, *Vestnik Leningrad. Univ. Mat. Mekh. Astronom.* **1** (1972), 15–25.
3. K. J. Föster and K. Petras, On the zeros of ultraspherical polynomials and Bessel functions, *in* "Orthogonal Polynomials and Their Applications" (C. Brezinski, L. Gori, and A. Ronveaux, Eds.), pp. 257–262, J. C. Baltzer AG, Scientific Publishing Co., ©IMACS, 1991.
4. L. Gatteschi, New inequalities for the zeroes of Jacobi polynomials, *SIAM J. Math. Anal.* **18** (1987), 1549–1562.
5. A. Guessab and G. V. Milovanovic, Weighted L^2-analogues of Bernstein's inequality and classical orthogonal polynomials, *J. Math. Anal. Appl.* **182** (1994), 244–249.
6. S. V. Konjagin, On estimates of derivatives of polynomials, *Dokl. Akad. Nauk SSSR* **243**(5) (1978), 1116–1118.
7. A. Lafogria and P. Siafarikas, Inequalities for the zeroes of ultraspherical polynomials, *in* "Orthogonal Polynomials and Their Applications" (C. Brezinski, L. Gori, and A. Ronveaux, Eds.), pp. 327–330, J. C. Baltzer AG, Scientific Publishing Co., ©IMACS, 1991.
8. G. Szegö, "Orthogonal Polynomials," revised edition, Vol. 23, Amer. Math. Soc., New York, 1959.
9. P. Turán, Remark on a theorem of Erhard Schmidt, *Mathematica* **2**(25) (1960), 373–378.